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Error of Approximation in Case of Definite Integrals

Rajesh Kumar Sinha, Satya Narayan Mahto, Dhananjay Sharan

Abstract— This paper proposes a method for computation of error of approximation involved in case of evaluation of integrals of single variable. The error associated with a quadrature rule provides information with a difference of approximation. In numerical integration, approximation of any function is done through polynomial of suitable degree but shows a difference in their integration integrated over some interval. Such difference is error of approximation. Sometime, it is difficult to evaluate the integral by analytical methods Numerical Integration or Numerical Quadrature can be an alternative approach to solve such problems. As in other numerical techniques, it often results in approximate solution. The Integration can be performed on a continuous function on set of data.

Index Terms— Quadrature rule, Simpsons rule, Chebyshev polynomials, approximation, interpolation, error.

1 INTRODUCTION

TO evaluate the definite integral of a function that has no explicit antiderivative of whose antiderivative is not easy to obtain; the basic method involved in approximating is numerical quadrature [1]-[4].

$$\int_a^b f(x)dx \tag{1}$$

i.e. $\sum_{i=0}^{n} \alpha_i f(x)$ to approximate $\int_a^b f(x)dx$. The methodology for computing the antiderivative at a given point, the polynomial $p(x)$ approximating the function $f(x)$ generally oscillates about the function. This means that if $y = p(x)$ over estimates the function $y = p(x)$ in one interval then it would underestimate it in the next interval [5]. As a result, while the area is overestimated in one interval, it may be underestimated in the next interval so that the overall effect of error in the two intervals will be equal to the sum of their moduli, instead the effect of the error in one interval will be neutralized to some extent by the error is the next interval. Therefore, the estimated error in an integration formula may be unrealistically too high. In view to above discussed facts, the paper would reveal types of approximation following the condition 'best' approximation for a given function, concentrating mainly on polynomial approximation. For approximation, there is considered a polynomial of first degree such as $y = a + bx$ a good approximation to a given function for the interval (a, b).

2 PROPOSED METHOD

2.1 Reflection on Approximation

This section cover types of approximation following the condition 'best' approximation for a given function, concentrating mainly on polynomial approximation. In this for approximation, there is considered a polynomial of first degree such as $y = a + bx$; a good approximation to a given continuous function for the interval (0, 1).

Under the assumption of given concept two following statements may be considered as,

The Taylor polynomial at $x = 0$ (assuming $f'(0)$ exists)

$$y = f(0) + xf'(0) \tag{2}$$

The interpolating polynomial constructed at $x = 0$ and $x = 1$.

$$y = f(0) + x[f(1) - f(0)] \tag{3}$$

A justification may be laid that a Taylor or interpolating polynomial constructed at some other point would be more suitable. However, these approximations are designed to initiate the behavior of f at only one or two points.

- Rajesh Kumar Sinha is with the Department of Mathematics, NIT Patna, India. E-mail: rajesh_nitpat@rediffmail.com
- Satya Narayan Mahto is with the Department of Mathematics, M. G. College, LNMU, Darbhanga, India.
- Dhananjay Sharan is research scholar in the Department of Mathematics, NIT Patna, India.

Since, the polynomial of first degree in x as shown above $y = a + bx$ follows a good approximation to f throughout the interval (0,1). Now, for values of a and b, the required mathematical exists such as $\max_{0 \le x \le 1} \left[f(x) = (a + bx) \right]$ is minimized over all choices of two values a and b. This expression is said as minimax (or Chebyshev) approximation. Instead of minimizing the minimum error between the (continuous) function f and the approximating straight-line, the process of maximizing 'sum' of the moduli of the errors may be undertaken.

For values of a and b, $\int_0^1 \left[f(x) = (a + bx) \right] dx$ is minimized that is called a base L_1 approximation. It should be noted that the L_1 approximation provides equal weight to all the errors, while the minimax approximation approximate in the error of largest modulus. Again, stressing on other approximation f which in a sense, lies between the extremes of L_1 and minimax approximation. Also, for a fixed value of $p \ge 1$, two values a and b are formed so that $\int_0^1 \left[f(x) - (a + bx) \right]^p dx$ is minimized and therefore would be suggested as best L_P approximation. The above maximized expression followed that due to the presence of the P^{th} power, the error of largest modulus tends to dominate as p increases with f continuous. It can be shown that, as $p \to \infty$ the best L_P approximation tends to the minimax approximation which is therefore sometimes called the best L_∞ approximation. Thus the L_P approximations consist of a spectrum ranging from the L_1 to the minimax approximations. Further, for $1 < p < \infty$, L_2 approximation, is the only commonly used and is better known as the best square approximation.

2.2 Generalized case for approximation under certain interval

This section would reveal for giving light on consideration of methods of approximating to f', given the values of $f(x)$ at certain points. If p is same polynomial, approximation to f an application of p' would exist an approximation to f'. However, there is need to be careful; the maximum modulus of $f'(x) - p'(x)$ on a given interval (a, b) can be much larger than the maximum modulus of $f(x) - p(x)$.

To make an evident proof for a given statement an assumption is made

$$f(x) - p(x) = 10^{-2} T_n(x) \tag{4}$$

Where $T_n(x)$ is a polynomial of degree n in x with leading term $2^{n-1} x^n$ for $n > 0$. The T_n are known as Chebyshev polynomials, the Russian mathematician P. L. Chebyshev (1821 -1894) who has contributed the equation (4) for the difference between of the first approximation and the polynomial. Now, to determine the turning value, of T_n the derivative

$$\frac{d}{dx} T_n(x) = \frac{d}{d\theta} \cos n\theta \frac{d\theta}{dx} = \frac{n \sin n\theta}{\sin\theta} \tag{5}$$

Since

$$\frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = -\frac{1}{\sin\theta} \tag{6}$$

$$\text{i.e. } T_n'(x) = \frac{n \sin n\theta}{\sin\theta} = n^2 \left(\frac{\sin n\theta}{n\theta} \right) \left(\frac{\theta}{\sin\theta} \right) \tag{7}$$

$$\text{but } \underset{\theta \to 0}{Lt} \left(\frac{\sin\theta}{\theta} \right) = 1 \tag{8}$$

$$\text{Thus } T_n'(1) = n^2 \tag{9}$$

It can be shown that this is the maximum modulus of T_n' on (-1, 1). If $n = 10$, say, the maximum modulus of $f - p = 10^{-2}$ on (-1, 1) whereas $f' - p' = 1$. Furthermore, the consideration of an approximation f and the polynomial p_n that interpolates the approximation f at distinct point $x_1 \ldots\ldots\ldots\ldots x_n$. If such happens then there exists a number ξ_x (depending on x) in certain interval (a, b) such that

$$f(x) - p_n(x) = (x - x_0)(x - x_1)\ldots\ldots\ldots\ldots(x - x_n) \frac{f^{(n+1)}(\xi_x)}{\lfloor (n+1)} \tag{10}$$

$$f(x) - p_n(x) = \pi_{n+1}(x) \frac{f^{(n+1)}(\xi_x)}{\lfloor (n+1)} \tag{11}$$

where $\pi_{n+1}(x) = (x - x_0)(x - x_1)\ldots\ldots\ldots(x - x_n)$.

Thus equation (11) is known as error that exists in following statement. If (a, b) is any interval and also contains (n +1) points $x, x_0, x_1, ...x_n$. Suppose further it is assumed that f, $f'.........f^n$ exist and are continuous on the interval (a, b) and $f^{(n+1)}$ exists for $a < x < b$ then the error holds as shown in equation (11). Thus, $f^{(n+1)}$ exists on some interval (a, b) which is $x, x_0, x_1, x_2, ..., x_n$. Here, the number $\xi_x \in (a,b)$ (depending on x). Differentiating (11) with respect to x,

$$f'(x) - p_n'(x) = \pi_{n+1}'(x) \frac{f^{(n+1)}(x)}{\underline{|(n+1)}} + \frac{\pi_{n+1}(x)}{\underline{|(n+1)}} \frac{d}{dx} f^{(n+1)}(\xi_x) \quad (12)$$

In general, there is nothing further to state about the second term on the right of equation (12). We can not perform the differentiation with respect to x of $f^{(n+1)}(\xi_x)$, since ξ_x is an unknown function of x. Thus, for Integral values of x given in equation (12) is unless for determining the accuracy with which p_n' approximately to f'. However, if we restrict x to one of the values $x_0, x_1, ...x_n$, then $\pi_{n+1}(x) = 0$ and the unknown second term on the right of equation (12) becomes zero.

$$f'(x_r) - p_n'(x) = \pi_{n+1}'(x_r) \frac{f^{(n+1)}(\xi_r)}{\underline{|(n+1)}} \quad (13)$$

Where ξ_r has been considered when $x = x_r$. Now, applying forward difference to express the polynomial $p_n(x)$ in given form such that

$$p_n(x) = p_n(x_0 + sh) \quad (14)$$

$$p_n(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + ... + \binom{s}{n}\Delta^n f_0 \quad (15)$$

Since $x = x_0 + sh$ (16)

i.e. $\frac{dx}{ds} = h$ (17)

Also

$$p_n(x) = p_n(x_0 + sh) \quad (18)$$

Differentiating both sides with respect to x,

$$p_n'(x) = \frac{ds}{dx} \frac{d}{ds} p_n(x_0 + sh) \quad (19)$$

$$p_n'(x) = \frac{1}{\frac{dx}{ds}} \left[\frac{d}{ds} \left\{ f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + ... + \binom{s}{n}\Delta^n f_0 \right\} \right] \quad (20)$$

$$p_n'(x) = \frac{1}{h} \left[\Delta f_0 + \frac{1}{2}(2s-1)\Delta^2 f_0 + ... + \frac{d}{ds}\binom{s}{n}\Delta^n f_0 \right] \quad (21)$$

To calculate π_{n+1}',

$$\pi_{n+1}(x) = (x - x_r) \prod_{j \neq r} (x - x_r) \quad (22)$$

where π_{n+1}' would be obtained by means of differentiating equations (22) such that

$$\pi_{n+1}'(x) = \left(\frac{d}{dx}(x - x_r) \right) \prod_{j \neq r} (x - x_j) + (x - x_r) \frac{d}{dx} \prod_{j \neq r} (x - x_r) \quad (23)$$

By putting $x = x_r$, the second term on the right of equation (22) becomes zero.

Thus $\pi_{n+1}'(x) = \prod_{j \neq r} (x - x_j) = (-1)^{n-r} h^n r!(n-r)!$ (24)

Since

$$x_r - x_j = (r - j)h \quad (25)$$

From (13)

$$f'(x_r) - p_n'(x_r) = (-1)^{(n-r)} h^n \frac{r!(n-r)!}{(n+r)!} f^{(n+1)}(\xi_r) \quad (26)$$

Furthermore, investigating the case of polynomial, an interpretation hold that polynomials are sufficiently accurate for many approximation and interpolation tasks.

2.3 Degree of Accuracy

Now the degree of accuracy of quadrature formula is the largest positive integer n such that the formula is exact for x^k, for each $k = 0,1,....n$. The Trapezoidal and Simpson's rule have degrees of precision one and three, respectively. Integration and summation are linear operations; that is,

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \qquad (27)$$

$$\sum_{i=0}^{n} (\alpha f(x_i) + \beta g(x_i)) = \alpha \sum_{i=0}^{n} f(x_i) + \beta \sum_{i=0}^{n} g(x_i) \qquad (28)$$

For each pair of integral functions f and g and each pair of real constants. This implies that the degree of precision of a quadrature formula in n if and only if the error $E(p(x)) = 0$ for all polynomials $p(x)$ of degree $k = 0,1,....n$, but $E(p(x)) \neq 0$ for some polynomial $p(x)$ of degree $(n+1)$.

3 Conclusion

Increasing, the degree of the approximating polynomial dose not guarantees better accuracy. In a higher degree polynomial, the coefficients also get bigger which may magnify the errors. Similarly, reducing the size of the sub-interval by increasing their number may also lead to accumulation of rounding errors. Therefore a balance should be kept between the two, i.e. degree of polynomial and total number of intervals. These are the primary motivations for studying the techniques of numerical integration/quadrature [6]-[9]. In case of Simpson's rule technique individually to the subintervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$; use error estimation procedure to determine if the approximation to the integral on subinterval is within a tolerance of $\varepsilon/2$. If so, then sum the approximations to procedure an approximation of function $f(x)$ over interval (a,b) within the tolerance ε. If the approximation on one of the subintervals fails to be within the tolerance $\varepsilon/2$, then that subinterval is itself subdivided, and the procedure is reapplied to the two subintervals to determine if the approximation on each subinterval is accurate to within $\varepsilon/4$. This halving procedure is continued until each portion is within the required tolerance. Thus, Numerical analysis is the study of algorithms that use numerical approximation for the problems of continuous functions [10]-[12]. Numerical analysis continues this long tradition of practical mathematical calculations. Much like the Babylonian approximation, modern numerical analysis does not seek exact answers, since the exact answers are often impossible to obtain in practice. Instead, much of numerical analysis is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors. It finds applications in all fields of engineering and the physical sciences.

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